The influence of Lebesgue functions on the convergence and summability of function series

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1. Introduction

Let X be a measurable space with a positive measure μ and let $\{f_n(x)\}$ be a sequence of μ-integrable functions on the measurable set $E\subset X$. Form the "Lebesgue functions"

$$L_n(x) = \int_E |K_n(t,x)|\,d\mu(t) \quad \text{and} \quad L_n^1(x) = \int_E |K_n^1(t,x)|\,d\mu(t),$$

where

$$K_n(t,x) = \sum_{k=0}^{n} f_k(t) f_k(x) \quad \text{and} \quad K_n^1(t,x) = \sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right) f_k(t) f_k(x).$$

These functions play an important role in the theory of convergence and summability of orthogonal series. We mention the following theorems of S. KACZMARZ [3], based on a method of A. KOLMOGOROFF—G. SELIVERSTOFF [4] and A. PLESSNER [5]:

A. If E is an interval of finite length on the real line, μ is the ordinary Lebesgue measure, and $\{f_n(x)\}$ is an orthonormal system defined on E, then the series $\Sigma a_n f_n(x)$ is convergent a.e. on E provided that $L_n(x)=O(\lambda_n)$ on E with $0<\lambda_n\leqq\lambda_{n+1}$ and $\Sigma a_n^2\lambda_n<\infty$.

B. Under the same conditions as above, $\Sigma a_n f_n(x)$ is $(C, 1)$-summable a.e. on E if, instead of $L_n(x)=O(\lambda_n)$, we only suppose $L_n^1(x)=O(\lambda_n)$ on E.

In the proof of these theorems the assumption that the system $\{f_n(x)\}$ is orthonormal was essentially exploited. Unexpectedly it turned out that *neither orthonormality nor L_μ^2-integrability of $\{f_n(x)\}$ is needed in theorems A and B.* It is enough to suppose that the functions $f_n(x)$ are L_μ-integrable on E and the condition

$$\int_E \left|\sum_{k=0}^{n} a_k b_k f_k(x)\right| d\mu(x) = O(1) \tag{1}$$

is satisfied whenever $\sum a_k^2 b_k^2 < \infty$.

We remark that (1) is trivially satisfied for orthonormal systems defined on a set E of finite measure, so that our results contain theorems A and B as special cases. Moreover, if we suppose $\lambda_n=1$ $(n=0, 1, \dots)$, i.e. if

$$L_n(x)=O(1) \quad \text{or} \quad L_n^1(x)=O(1) \tag{2}$$

on E, then even (1) is unnecessary. Hence *if one of the conditions* (2) *is uniformly valid on E, then* $\Sigma a_n f_n(x)$ *is a. e. convergent or* $(C, 1)$*-summable on E, respectively, under the sole condition* $\Sigma a_n^2<\infty$. So we can say that some classical theorems as for instance the theorem of Fejér—Lebesgue applied to the Fourier series of L^2-integrable functions is but a special case of our theorem belonging to the general theory of real functions.

As to the proof, we proceeded originally on the same way we followed in the case of multiplicatively orthogonal series (see [2]). C. I. PRESTON, after having read a preprint of [2], has communicated in a letter to the first author an idea which simplified also a part of our original proof very much. (The note of DR. PRESTON referring to this will appear later*). In the present paper we shall use his idea in the proof of Theorems 1 and 5.

2. The convergence problem

Let $\{f_n(x)\}$ be a sequence of L_μ-integrable functions on the μ-measurable set E and $\{\lambda_n\}$ a non-decreasing sequence of positive numbers. Denote further by $s_n(x)$ the n-th partial sum $\sum_{k=0}^{n} a_k f_k(x)$ of the series $\sum_{n=0}^{\infty} a_n f_n(x)$.

Theorem 1. *If* $\Sigma a_n^2<\infty$ *and the Lebesgue functions* $L_{\nu_n}(x)$ *satisfy the condition*

$$L_{\nu_n}(x)=O(\lambda_{\nu_n})$$

uniformly on the measurable set E of finite measure, then $s_{\nu_n}(x)=O_x(\lambda_{\nu_n}^{\frac{1}{2}})$ *on E almost everywhere.*

Denote by $n(x)$ the least index m $(\leqq n)$ for which

$$\lambda_{\nu_m}^{-\frac{1}{2}} s_{\nu_m}(x) = \max_{0\leqq k\leqq n} \lambda_{\nu_k}^{-\frac{1}{2}} s_{\nu_k}(x).$$

We proceed to prove that the left hand side is finite on E a. e. For this purpose we use an idea of Preston which consists in a special representation of $s_{\nu_{n(x)}}(x)$.

*) Meanwhile it was published in the *J. Amer. Math. Soc.*, **28** (1971), 453—455.

Introduce an arbitrary orthonormal system $\{g_k(y)\}$ defined on a measure space Y with positive measure ν, then

$$\lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x) = \int_Y \sum_{k=0}^{\nu_n} a_k g_k(t) \cdot \lambda_{\nu_{n(x)}}^{-\frac{1}{2}} \sum_{k=0}^{\nu_{n(x)}} g_k(t) f_k(x)\, d\nu(t).$$

So we obtain by Schwarz's inequality

$$\left| \int_E \lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x)\, d\mu(x) \right| \leqq$$

$$\leqq \left\{ \int_Y \left[\sum_{k=0}^{\nu_n} a_k g_k(t) \right]^2 d\nu(t) \cdot \int_Y \left[\int_E \lambda_{\nu(n)x}^{-\frac{1}{2}} \sum_{k=0}^{\nu_{n(x)}} g_k(t) f_k(x)\, d\mu(x) \right]^2 d\nu(t) \right\}^{\frac{1}{2}} \leqq$$

$$\leqq \left\{ \sum_{k=0}^{\infty} a_k^2 \right\}^{\frac{1}{2}} \left\{ \int_E \int_E \int_Y \lambda_{\nu_{n(x)}}^{-\frac{1}{2}} \sum_{k=0}^{\nu_{n(x)}} g_k(t) f_k(x) \cdot \lambda_{\nu_{n(y)}}^{-\frac{1}{2}} \sum_{k=0}^{\nu_{n(y)}} g_k(t) f_k(y)\, d\mu(x)\, d\mu(y)\, d\nu(t) \right\}^{\frac{1}{2}} =$$

$$= O(1) \left\{ \int_E \int_E \lambda_{\nu_n(x)}^{-\frac{1}{2}} \lambda_{\nu_n(y)}^{-\frac{1}{2}} \left| \sum_{k=0}^{\nu_{n(x,y)}} f_k(x) f_k(y) \right| d\mu(x)\, d\mu(y) \right\}^{\frac{1}{2}},$$

where $n(x, y) = \min\{n(x), n(y)\}$. As the sum in the last integrand equals $K_{\nu_{n(x,y)}}(x, y)$, it follows

$$\left| \int_E \lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x)\, d\mu(x) \right| =$$

$$= O(1) \left\{ \int_E \int_E \lambda_{\nu_{n(x)}}^{-1} |K_{\nu_{n(x)}}(x, y)|\, d\mu(x)\, d\mu(y) + \int_E \int_E \lambda_{\nu_{n(y)}}^{-1} |K_{\nu_{n(y)}}(x, y)|\, d\mu(x)\, d\mu(y) \right\}^{\frac{1}{2}} =$$

$$= O(1) \left\{ \int_E \lambda_{\nu_{n(x)}}^{-1} L_{\nu_{n(x)}}(x)\, d\mu(x) + \int_E \lambda_{\nu_{n(y)}}^{-1} L_{\nu_{n(y)}}(y)\, d\mu(y) \right\}^{\frac{1}{2}} = O(1).$$

Since the sequence $\{\lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x)\}$ is increasing, it follows by B. Levi's theorem that

$$\lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x) < \infty \quad \text{a.e.}$$

The same is true for the sequence $\{-\lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x)\}$; hence

$$\lambda_{\nu_{n(x)}}^{-\frac{1}{2}} |s_{\nu_{n(x)}}(x)| = O_x(1) \qquad \text{a.e,}$$

which contains our statement.

Theorem 2. *If the Lebesgue functions $L_n(x)$ are uniformly bounded on the measurable set E of finite measure and $\Sigma a_n^2 < \infty$, then the series $\Sigma a_n f_n(x)$ is convergent on E a. e.*

Indeed, $\Sigma a_n^2<\infty$ implies $\Sigma a_n^2\mu_n<\infty$ with an appropriate increasing sequence $\{\mu_n\}$ of positive numbers tending to infinity. Then we get by partial summation for every $m\geqq n$:

$$|s_m(x)-s_n(x)| = \left|\sum_{k=n+1}^{m} \mu_k^{-\frac{1}{2}}\mu_k^{\frac{1}{2}} a_k f_k(x)\right| \leqq$$

$$\leqq \sum_{k=n+1}^{m-1} (\mu_k^{-\frac{1}{2}}-\mu_{k+1}^{-\frac{1}{2}})\left|\sum_{l=0}^{k}\mu_l^{\frac{1}{2}} a_l f_l(x)\right| + \mu_{n+1}^{-\frac{1}{2}}\left|\sum_{l=0}^{n}\mu_l^{\frac{1}{2}} a_l f_l(x)\right| + \mu_m^{-\frac{1}{2}}\left|\sum_{l=0}^{m}\mu_l^{\frac{1}{2}} a_l f_l(x)\right|.$$

Since $\Sigma a_n^2\mu_n<\infty$ and $L_n(x)=O(1)$ for $x\in E$, we can apply Theorem 1 with $v_n=n$ and $\lambda_n=1$ for every n. It follows then

$$\sum_{l=0}^{k}\mu_l^{\frac{1}{2}} a_l f_l(x)=O_x(1)$$

for every k and almost all $x\in E$; hence $s_m(x)-s_n(x) = o_x(1)$ a.e.

Theorem 3. *Suppose $L_{v_n}(x)=O(\lambda_{v_n})$ for every $x\in E$ and $\Sigma a_n^2\lambda_n<\infty$. If also condition* (1) *is satisfied, then the sequence $\{s_{v_n}(x)\}$ of partial sums of the series $\Sigma a_n f_n(x)$ converges on E a.e.*

Set

$$S_n(x) = \sum_{k=0}^{n}\lambda_k^{\frac{1}{2}}\mu_k^{\frac{1}{2}} a_k f_k(x)$$

with an appropriate increasing sequence $\{\mu_n\}$ of positive numbers tending to infinity, and $\Sigma a_n^2\lambda_n\mu_n<\infty$. We proceed as in the proof of Theorem 2:

$$|s_{v_m}(x)-s_{v_n}(x)| \leqq \sum_{k=v_n+1}^{v_m-1} [(\lambda_k\mu_k)^{-\frac{1}{2}}-(\lambda_{k+1}\mu_{k+1})^{-\frac{1}{2}}]|S_k(x)| + \tag{3}$$

$$+(\lambda_{v_n+1}\mu_{v_n+1})^{-\frac{1}{2}}|S_{v_n}(x)|+(\lambda_{v_m}\mu_{v_m})^{-\frac{1}{2}}|S_{v_m}(x)|.$$

Because of $\Sigma a_k^2\lambda_k\mu_k<\infty$ we have by Theorem 1

$$S_{v_n}(x)=O_x(\lambda_{v_n}^{\frac{1}{2}}) \quad \text{and} \quad S_{v_m}(x)=O_x(\mu_{v_m}^{\frac{1}{2}}) \quad \text{a.e.}$$

So the last two terms in (3) have the order of magnitude $o_x(1)$ a.e. on E. Regarding the first sum on the right hand side consider the series

$$S = \sum_{k=0}^{\infty}[(\lambda_k\mu_k)^{-\frac{1}{2}}-(\lambda_{k+1}\mu_{k+1})^{-\frac{1}{2}}]\int_E |S_k(x)|\, d\mu(x).$$

Apply condition (1) with $b_k=(\lambda_k\mu_k)^{\frac{1}{2}}$, then the integrals on the right hand side are of order $O(1)$, hence

$$S = O(1)\sum_{k=0}^{\infty}[(\lambda_k\mu_k)^{-\frac{1}{2}}-(\lambda_{k+1}\mu_{k+1})^{-\frac{1}{2}}] < \infty.$$

B. Levi's theorem implies the convergence a.e. of the series

$$\sum_{k=0}^{\infty}[(\lambda_k\mu_k)^{-\frac{1}{2}}-(\lambda_{k+1}\mu_{k+1})^{-\frac{1}{2}}]S_k(x),$$

so we get from (3) $s_{\nu_m}(x)-s_{\nu_n}(x) = o_x(1)$ on E a. e.

Remark. Condition (1) can be weakened. We chose it only to get a simple and clear form of Theorem 3. But it could be replaced e.g. by

$$\int_E |S_k(x)|\,d\mu(x) = O\left(\lambda_k^{\frac{1}{2}-\varepsilon}\mu_k\right) \qquad (\varepsilon>0)$$

supposing also that $\{1/\lambda_n\}$ is convex. It is easy to see that the series S would converge also under this condition.

As application of Theorem 3 we prove one of our results concerning multiplicatively orthogonal series, [2]. A system $\{\varphi_n(x)\}$ is called multiplicatively orthogonal on the measurable set E, if every finite product of different φ_k's has zero integral on E. That is, setting the product system $\psi_0(y)\equiv 1$ and $\psi_n(x)=\varphi_{m_1+1}(x)\varphi_{m_2+1}(x)\ldots\varphi_{m_k+1}(x)$ for $n = 2^{m_1}+2^{m_2}+\cdots+2^{m_k}$, we have

$$\int_E \psi_n(x)\,d\mu(x)=0 \qquad (n\geqq 1).$$

Theorem 4. *Let $\{\varphi_n(x)\}$ be a multiplicatively orthogonal system defined on a measurable set E of finite measure. If $|\varphi_n(x)|\leqq M_n$, then $\Sigma c_n^2 M_n^2<\infty$ implies the convergence a.e. on E of the series $\Sigma c_n\varphi_n(x)$.*

Denote by $\{f_n(x)\}$ the above defined product system of $\{\varphi_n(x)/M_n\}$ and set $a_n=c_{\nu+1}M_{\nu+1}$ for $n=2^\nu$, and $a_n=0$ for $n\neq 2^\nu$. Then we may write

$$s_{2^n-1}(x) = \sum_{k=0}^{2^n-1} a_k f_k(x) = \sum_{k=1}^{n} c_k M_k\cdot\frac{1}{M_k}\varphi_k(x).$$

We apply Theorem 3 with $\nu_n = 2^n-1$ and $\lambda_n=1$ $(n=0, 1, \ldots)$. The Lebesgue functions $L_{2^n-1}(x)$ of the system $\{f_n(x)\}$ defined in this way are uniformly bounded on E, because

$$K_{2^n-1}(t, x) = \sum_{k=0}^{2^n-1} f_k(t)f_k(x) = \prod_{k=1}^{n}\left(1+\frac{\varphi_k(t)\varphi_k(x)}{M_k^2}\right)\geqq 0,$$

and hence

$$L_{2^n-1}(x) = \int_E \sum_{k=0}^{2^n-1} f_k(t)f_k(x)\,d\mu(t) = \int_E d\mu(t).$$

Thus we have only to show that (1) is also satisfied. Choose, for this aim, $\{b_k\}$

arbitrarily so that $\Sigma a_k^2 b_k^2 < \infty$. Then, for every n of the form 2^m+p with $0 \leqq p < 2^m$ we have

$$\int_E \left| \sum_{k=0}^{n} a_k b_k f_k(x) \right| d\mu(x) = \left| \int_E \sum_{\nu=0}^{m} c_{\nu+1} b_{2^\nu} \cdot \frac{1}{M_{\nu+1}} \varphi_{\nu+1}(x)\, d\mu(x) \right| \leqq$$

$$\leqq \left\{ \int_E d\mu(x) \int_E \left[\sum_{\nu=0}^{m} c_{\nu+1} b_{2^\nu} \cdot \frac{1}{M_{\nu+1}} \varphi_{\nu+1}(x) \right]^2 d\mu(x) \right\}^{\frac{1}{2}} =$$

$$= O(1) \left\{ \sum_{\nu=0}^{m} c_{\nu+1}^2 b_{2^\nu}^2 \right\}^{\frac{1}{2}} = O(1) \left\{ \sum_{k=0}^{\infty} a_k^2 b_k^2 \right\}^{\frac{1}{2}} = O(1).$$

Hence condition (1) is satisfied and our statement proved.

3. The summation problem

Theorem 5. *If $\Sigma a_k^2 < \infty$ and the Lebesgue functions $L_n^1(x)$ satisfy the condition $L_n^1(x) = O(\lambda_n)$ uniformly on the measurable set E of finite measure, then the sums*

$$\sigma_n(x) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k f_k(x)$$

have the order of magnitude $O_x(\lambda_n^{\frac{1}{2}})$ on E, a.e.

Denote by n_x the least index m ($\leqq n$) for which

$$\lambda_m^{-\frac{1}{2}} \sigma_m(x) = \max_{0 \leqq k \leqq n} \lambda_k^{-\frac{1}{2}} \sigma_k(x),$$

and set $n_{x,y} = \min(n_x, n_y)$. Let be $\{g_k(y)\}$ an arbitrary orthonormal system defined in a measure space (Y, ν). Then

$$\left| \int_E \lambda_{n_x}^{-\frac{1}{2}} \sigma_{n_x}(x)\, d\mu(x) \right| = \left| \int_E \int_Y \sum_{k=0}^{n} a_k g_k(t) \cdot \lambda_{n_x}^{-\frac{1}{2}} \sum_{k=0}^{n_x} \left(1 - \frac{k}{n_x+1}\right) g_k(t) f_k(x)\, d\nu(t)\, d\mu(x) \right| \leqq$$

$$\leqq \left\{ \int_Y \left[\sum_{k=0}^{n} a_k g_k(t) \right]^2 d\nu(t) \int_E \int_E \int_Y \lambda_{n_x}^{-\frac{1}{2}} \lambda_{n_y}^{-\frac{1}{2}} \sum_{k=0}^{n_x} \left(1 - \frac{k}{n_x+1}\right) g_k(t) f_k(x) \times \right.$$

$$\left. \times \sum_{k=0}^{n_y} \left(1 - \frac{k}{n_y+1}\right) g_k(t) f_k(x)\, d\mu(x)\, d\mu(y)\, d\nu(t) \right\}^{\frac{1}{2}} =$$

$$= O(1) \left\{ \int_E \int_E \lambda_{n_x}^{-\frac{1}{2}} \lambda_{n_y}^{-\frac{1}{2}} \sum_{k=0}^{n_{x,y}} \left(1 - \frac{k}{n_x+1}\right) \left(1 - \frac{k}{n_y+1}\right) f_k(x) f_k(y)\, d\mu(x)\, d\mu(y) \right\}^{\frac{1}{2}}.$$

Apply twice partial summation to the sum in the last integrand to get

$$\sum_{k=0}^{n_{x,y}}\left(\frac{1}{n_x+1}+\frac{1}{n_y+1}-\frac{2k+1}{(n_x+1)(n_y+1)}\right)K_k(x,y)=$$

$$=\frac{2}{(n_x+1)(n_y+1)}\sum_{k=0}^{n_{x,y}-1}(k+1)K_k^1(x,y)+$$

$$+n_{x,y}K^1_{n_{x,y}}(x,y)\left(\frac{1}{n_x+1}+\frac{1}{n_y+1}-\frac{2n_{x,y}-1}{(n_x+1)(n_y+1)}\right);$$

hence it follows

$$\left|\int_E \lambda_{n_x}^{-\frac{1}{2}}\sigma_{n_x}(x)\,d\mu(x)\right|=$$

$$=O(1)\left\{\int_E\int_E \lambda_{n_x}^{-1}\left[(n_x+1)^{-2}\sum_{k=0}^{n_x}(k+1)|K_k^1(x,y)|+|K^1_{n_x}(x,y)|\right]d\mu(x)\,d\mu(y)\right\}^{\frac{1}{2}}=$$

$$=O(1)\left\{\int_E (n_x+1)^{-2}\sum_{k=0}^{n_x}(k+1)\lambda_{n_x}^{-1}L_k^1(x)\,d\mu(x)+\int_E \lambda_{n_x}^{-1}L_{n_x}(x)\,d\mu(x)\right\}^{\frac{1}{2}}=O(1).$$

Reasoning as before in the proof of Theorem 1 we obtain

$$|\sigma_{n_x}(x)|=O_x(\lambda_{n_n}^{\frac{1}{2}})\quad\text{a.e.}$$

Theorem 6. *If the Lebesgue functions $L_n^1(x)$ are uniformly bounded on the measurable set E of finite measure and $\Sigma a_n^2<\infty$, then the series $\Sigma a_n f_n(x)$ is $(C,1)$-summable a.e.*

The convergence of Σa_n^2 implies the existence of a sequence $\{\mu_n\}$ of positive numbers, concave from below and tending to infinity such that $\Sigma a_n^2\mu_n<\infty$. Denote by $\Delta\mu_n^{-\frac{1}{2}}$ and $\Delta^2\mu_n^{-\frac{1}{2}}$ the first and the second differences of $\{\mu_n^{-\frac{1}{2}}\}$, respectively. Put $\sigma_n(\sqrt{\mu},x)$ the n-th $(C,1)$ mean of the series $\Sigma a_n\sqrt{\mu_n}f_n(x)$. By a known identity (see e.g. [1], p. 72) we have

$$\sigma_m(x)-\sigma_n(x)=\sum_{k=0}^{m-1}\left(1-\frac{k}{m+1}\right)\Delta^2\mu_k^{-\frac{1}{2}}\cdot(k+1)\sigma_k(\sqrt{\mu},x)- \tag{4}$$

$$-\sum_{k=0}^{n-1}\left(1-\frac{k}{n+1}\right)\Delta^2\mu_k^{-\frac{1}{2}}\cdot(k+1)\sigma_k(\sqrt{\mu},x)+\frac{2}{m+1}\sum_{k=0}^{m-1}\Delta\mu_{k+1}^{-\frac{1}{2}}(k+1)\sigma_k(\sqrt{\mu},x)-$$

$$-\frac{2}{n+1}\sum_{k=0}^{n-1}\Delta\mu_{k+1}^{-\frac{1}{2}}\cdot(k+1)\sigma_k(\sqrt{\mu},x)+\mu_m^{-\frac{1}{2}}\sigma_m(\sqrt{\mu},x)-\mu_n^{-\frac{1}{2}}\sigma_n(\sqrt{\mu},x).$$

From Theorem 5 we get $\sigma_k(\sqrt{\mu},x)=O_x(1)$ a.e.; hence the last two terms in (4) have the order of magnitude $o_x(1)$ a.e. Since the sequence $\{\mu_n^{-\frac{1}{2}}\}$ is convex and tends to zero, it follows $\Delta\mu_n^{-\frac{1}{2}}=o(n^{-1})$. Thus the third and fourth terms in (4) are also $o_x(1)$ a.e. In what concerns the first two terms, take into account that by Theorem 5 the series

$$\sum_{k=0}^{\infty} \Delta^2 \mu_k^{-\frac{1}{2}} \cdot (k+1) \left|\sigma_k(\sqrt{\mu}, x)\right| = O_x(1) \sum_{k=0}^{\infty} (k+1) \Delta^2 \mu_k^{-\frac{1}{2}}$$

converges a. e. because of the convexity of $\{\mu_k^{-\frac{1}{2}}\}$. The first two terms in (4), being the difference of the m-th and n-th $(C, 1)$ means of an a. e. convergent series, tend to zero a.e., consequently

$$\sigma_m(x) - \sigma_n(x) = o_x(1) \quad \text{a.e.} \quad (m > n).$$

Theorem 7. *Let $\{\lambda_n\}$ be an increasing sequence of positive numbers concave from below. Suppose $L_n^1(x) = O(\lambda_n)$ for every $x \in E$ and $\Sigma a_n^2 \lambda_n < \infty$. If condition* (1) *is also satisfied, then the series $\Sigma a_n f_n(x)$ is $(C, 1)$-summable on E almost everywhere.*

Choose first a sequence $\{\mu_n\}$ of positive numbers concave from below and tending to infinity, such that $\Sigma a_n^2 \lambda_n \mu_n < \infty$ and that $\{\lambda_n \mu_n\}$ be concave from below. Using the same notations as in the proof of Theorem 6, with $\lambda_n \mu_n$ instead of μ_n, we get first

$$\sigma_n(x) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n+1}\right) \Delta^2 (\lambda_k \mu_k)^{-\frac{1}{2}} (k+1) \sigma_k(\sqrt{\lambda \mu}, x) + \tag{5}$$

$$+ \frac{2}{n+1} \sum_{k=0}^{n-1} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} (k+1) \sigma_k(\sqrt{\lambda \mu}, x) + (\lambda_n \mu_n)^{-\frac{1}{2}} \sigma_n(\sqrt{\lambda \mu}, x).$$

By Theorem 5 we have $\sigma_n(\sqrt{\lambda \mu}, x) = O_x(\lambda_n^{-\frac{1}{2}})$ a.e.; hence the last term on the right hand side is $o_x(1)$ a. e. As to the second, it follows by condition (1):

$$\sum_{k=0}^{\infty} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} \int_E \left|\sigma_k(\sqrt{\lambda \mu}, x)\right| d\mu(x) < \infty,$$

thus $\Sigma \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} \left|\sigma_k(\sqrt{\lambda \mu}, x)\right|$ converges a. e. This implies the existence of an index $N = N(x, \varepsilon)$ such that for an arbitrary $\varepsilon > 0$

$$\sum_{k=N}^{\infty} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} \left|\sigma_k(\sqrt{\lambda \mu}, x)\right| < \frac{\varepsilon}{4} \quad \text{a.e.}$$

Therefore we get for sufficiently large n and almost all $x \in E$

$$\left| \frac{2}{n+1} \sum_{k=0}^{n-1} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} (k+1) \sigma_k(\sqrt{\lambda \mu}, x) \right| \leqq$$

$$\leqq \frac{2}{n+1} \sum_{k=0}^{N-1} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} (k+1) \left|\sigma_k(\sqrt{\lambda \mu}, x)\right| +$$

$$+ 2 \sum_{k=N}^{n-1} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} \left|\sigma_k(\sqrt{\lambda \mu}, x)\right| = O_x(1) \frac{N^2}{n+1} + \frac{\varepsilon}{2} < \varepsilon.$$

In other words, the second term on the right hand side of (5) is $o_x(1)$ on E a.e.. Concerning the first term we get

$$\sum_{k=0}^{\infty} \Delta^2(\lambda_k\mu_k)^{-\frac{1}{2}}(k+1)\int_E \left|\sigma_k(\sqrt{\lambda\mu}, x)\right| d\mu(x) < \infty$$

by condtion (1) and the convexity of $\{(\lambda_k\mu_k)^{-\frac{1}{2}}\}$, i. e. the series

$$\Sigma\Delta^2(\lambda_k\mu_k)^{-\frac{1}{2}}(k+1)\sigma_k(\sqrt{\lambda\mu}, x)$$

converges a. e. The first term, being about the $(n-1)$th $(C, 1)$ mean of this series, is also convergent for almost all $x\in E$. Thus we see by (5) that $\{\sigma_n(x)\}$ is decomposed for almost all $x\in E$ in a convergent sequence and two terms of order $o_x(1)$. Consequently, $\{\sigma_n(x)\}$ converges a.e. as we have stated.

Remarks. 1. One can easily see that Theorems 2 and 6 cannot be improved. Indeed, *if $a_0, a_1, \ldots$ are arbitrary real numbers such that $\Sigma a_n^2=\infty$, there exists a system $\{f_n(x)\}$ of continuous functions in* (0, 1) *with uniformly bounded Lebesgue functions such that $\Sigma a_n f_n(x)$ is nowhere summable by any regular positive Toeplitz method.*

Choose $E=[0, 1]$ with the ordinary Lebesgue measure μ. The system

$$f_n(x)=a_n\left(\sum_{\nu=0}^{n} a_\nu^2\right)^{-1} \qquad (0\leqq x\leqq 1)$$

has the required property. Indeed, for every $x\in[0, 1]$ we have

$$L_n(x) = \int_0^1 \sum_{k=0}^{n} a_k^2\left(\sum_{\nu=0}^{k} a_\nu^2\right)^{-2} dt \leqq \sum_{k=0}^{\infty} a_k^2\left(\sum_{\nu=0}^{k} a_\nu^2\right)^{-2}.$$

The last series is convergent, and hence $L_n(x)=O(1)$ uniformly in [0, 1]. But

$$s_n(x) = \sum_{k=0}^{n} a_k^2\left(\sum_{\nu=0}^{k} a_\nu^2\right)^{-1} \to \infty.$$

The terms of $\Sigma a_n f_n(x)$ are positive, thus the series is not summable by any regular positive Toeplitz method.

2. Theorems 5, 6, and 7 can be generalized in that way that their statements remain valid for any (C, α)-summation $(\alpha>0)$, if we substitute $L_n^1(x)$ by the corresponding Lebesgue functions $L_n^\alpha(x)$. The proofs are similar, but longer, because of the more intricate computations with (C, α) means. The technique could be copied from [6].

3. One can see that condition (1) could be weakened also in Theorem 7 exactly as we indicated in our remark to Theorem 3. Moreover we could substitute the condition of concavity of the sequence $\{\lambda_n\}$ by other, somewhat less pretentious conditions. But it seemed us that a simple form of Theorem 7 shows clearer the essence than any other more sophisticated statement.

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